Vibrations of Constrained Cylindrical Shells

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A study is made of the vibration characteristics of a cylindrical shell with arbitrary boundary conditions and with several intermediate constraint positions between the ends. The solution is obtained using a Rayleigh-Ritz procedure in which the axial modal displacements are constructed from simple Fourier series expressions. Geometric boundary conditions that are not identically satisfied are enforced with Lagrange multipliers. Unwanted geometric boundary conditions, forced to be zero due to the nature of the assumed series, are released through the mechanism of Stokes' transformation. For the problem without intermediate constraint, comparison with other investigators yields excellent agreement. For the problem with intermediate constraint, results are presented for a wide variety of constraint positions and types, boundary conditions and circumferential mode numbers.

Nomenclature

h,R,ℓ	= thickness, radius, and length of shell
k	= $h^2/12R^2$
m,n	= Fourier component in longitudinal and circumferential directions
u,v,w	= axial, circumferential, and radial displacement
x,θ	= axial and circumferential coordinates
t	= time
A_{mn},B_{mn},C_{mn}	= Fourier coefficients for $u,v,$ and w
D_i	= unspecified end value
D	= extensional rigidity $Eh/(1-\nu^2)$
E	= Young's modulus
G	= shear modulus
K	= flexural rigidity $Eh^3/12(1-\nu^2)$
$M_x,M_\theta,\bar{M}_{x\theta}$	= bending and twisting moment per unit length
$N_x,N_\theta,\bar{N}_{x\theta}$	= membrance force per unit length
$\hat{N}_{x\theta}$	= effective membrance shear force per unit length
$\hat{Q}_x$	= effective transverse shear force per unit length
T,U	= kinetic energy, strain energy
$\alpha_i,\beta_i,\gamma_i,\lambda_i$	= Lagrange multipliers
ξ_i	= intermediate constraint position
δ_i	= dimensionless, intermediate constraint position ($=\xi_i/\ell$)
ν	= Poisson's ratio
ρ	= mass density
ψ_u,ψ_v,ψ_w	= axial mode shapes corresponding to axial circumferential, and radial directions, respectively
ω	= circular frequency of shell
ω_0	= lowest extensional frequency of a ring in plane strain ($=\sqrt{E}/\{\rho R^2(1-\nu^2)\}$)
Ω	= frequency parameter ($=\omega^2/\omega_0{}^2$)
$(\ldots)_{,x},(\ldots)'$	= $\partial(\;)/\partial x$, $\partial(\;)/\partial x$
$(\ldots)_{,t},(\ldots)_\theta$	= $\partial(\;)/\partial t$, $\partial(\;)/\partial\theta$

Received December 17, 1974; revision received March 19, 1975.
Index category: Structural Dynamic Analysis.

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Introduction

THE vibratory characteristics of thin shells have been the subject of intense study in recent years. Investigations have been performed for many different boundary conditions and the results have been tabulated. This paper is a study of the vibration characteristics of a cylindrical shell with arbitrary boundary conditions and with several arbitrary intermediate constraint conditions between the ends. Applications for shells of this type abound in industry. Often, a shell designed and constructed for a certain application has to be suitably altered (constrained) to change its vibratory behavior. A systematic study is made here of these problems using the Rayleigh-Ritz procedure in conjunction with Lagrange multipliers. This technique had been used previously for a variety of buckling and vibration problems.[1] Recently, Dowell[2,3] has extended this technique and investigated problem areas such as the vibration of stiffened plates. In another recent work, the present authors used this Rayleigh-Ritz procedure with Lagrange multipliers to solve a constrained shell problem associated only with the axisymmetric (breathing) mode of vibration for one simplified set of boundary conditions.[4]

The problem is treated within the framework of linear, elastic, first-order shell theory. Starting with Sanders' theory,[5] the dependent variables are expanded into Fourier series in the *circumferential* direction. The problem is then considered on a mode by mode basis associated with the circumferential Fourier index n. The *axial* dependence for the constrained shell is constructed from the axial modal displacements of a suitably unconstrained shell. These modes, in the form of simple trigonometric series, are built up to form complete orthogonal sets, which are not restricted to term by term satisfaction of boundary conditions. If these sets do not satisfy the geometric boundary conditions of the constrained shell, then Lagrange multipliers are used to enforce these conditions. Similarly, Lagrange multipliers are used to satisfy conditions at the intermediate constraints. Due to the nature of the Rayleigh-Ritz energy method, it is not necessary to give explicit consideration to the natural boundary conditions. On the other hand, sometimes certain geometric boundary conditions are forced to be zero due to the nature of the assumed displacement functions. If these boundary conditions are unwanted, they may be released through the mechanism of Stokes' transformation.[6] To the best of the authors' knowledge, this use of Stokes' transformation has never been considered before in the solution of shell problems.

Theoretical Background

The circular cylindrical shell is shown in Fig. 1. Also shown are the forces and moments acting on a section of the shell parallel to the coordinate lines.

Rayleigh-Ritz Procedure

In the Rayleigh-Ritz method it is essential to obtain expressions for the strain energy U and the kinetic energy T. Following Sanders' theory[5] these expressions may be written entirely in terms of modal displacements and their derivatives

$$U=\frac{Eh}{2R(1-\nu^2)}\int_0^{2\pi}\int_0^{\ell}[R^2u_{,x}^2+(v_{,\theta}+w)^2$$
$$+2\nu Ru_{,x}(v_{,\theta}+w)+\frac{1-\nu}{2}(u_{,\theta}+Rv_{,x})^2+k\{R^4w_{,xx}^2$$
$$+(w_{,\theta\theta}-v_{,\theta})^2+2\nu R^2w_{,xx}(w_{,\theta\theta}-v_{,\theta})$$
$$+\frac{1-\nu}{8}(u_{,\theta}-3Rv_{,x})^2+2(1-\nu)R^2w_{,x\theta}^2$$
$$+(1-\nu)Ru_{,\theta}w_{,x\theta}-3(1-\nu)R^2v_{,x}w_{,x\theta}\}]\,dxd\theta \qquad (1)$$

$$T=\tfrac{1}{2}\rho h\int_0^{2\pi}\int_0^{\ell}(u_{,t}^2+v_{,t}^2+w_{,t}^2)R\,dxd\theta \qquad (2)$$

In the Rayleigh-Ritz procedure to be used, series expressions are chosen to represent the displacements. Then the variational principle $\delta(U_{\max}-T_{\max})=0$ is employed where $U_{\max}$ and $T_{\max}$ are the maximum (in time) of the strain energy and the kinetic energy, respectively. The variation is with respect to all displacements that do not violate geometric boundary conditions. It is perhaps worth noting at this point that it is perfectly permissible to set up this vibration problem in terms of Lagrange's equations.[2,3] However, since we know in advance that the generalized coordinates will vary sinusoidally, the present method seems more direct.

The boundary conditions applicable to Sanders' theory reduce to prescribing the following quantities at the ends of the cylindrical shell.

$$N_x \text{ or } u,\ \hat{N}_{x\theta} \text{ or } v,\ \hat{Q}_x \text{ or } w,\ M_x \text{ or } \partial w/\partial x \quad x=0,\ell \qquad (3)$$

where

$$\hat{N}_{x\theta}=\bar{N}_{x\theta}+\frac{3}{2R}\bar{M}_{x\theta},\quad \hat{Q}_x=\frac{\partial M_x}{\partial x}+\frac{2}{R}\frac{\partial \bar{M}_{x\theta}}{\partial\theta} \qquad (4)$$

The effective forces per unit length $\hat{N}_x$ and $\hat{Q}_x$ are shown in Fig. 1. The force quantities on the left of Eq. (3) are associated with the natural boundary conditions while the displacement quantities on the right are associated with the geometric boundary conditions. Further information on these relationships may be found in Refs. (5) and (7). It will prove useful to have the relationships between the boundary forces and displacements explicitly written out

$$N_x=D[u_{,x}+(\nu/R)v_{,\theta}+(\nu/R)w] \qquad (5a)$$

$$\hat{N}_{x\theta}=\frac{D(1-\nu)}{2}\left[\frac{1}{R}\left(1-\frac{3k}{4}\right)u_{,\theta}\right.$$
$$\left.+\left(1+\frac{9k}{4}\right)v_{,x}-3kw_{,x\theta}\right] \qquad (5b)$$

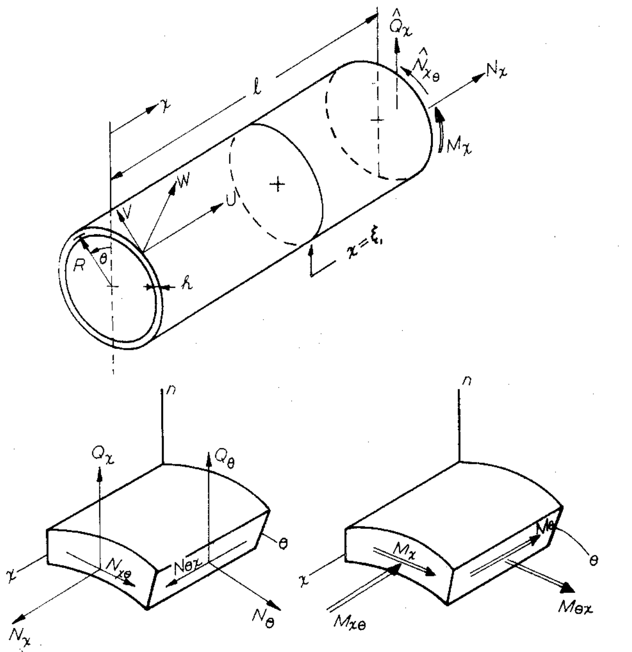

Fig. 1 Cylindrical shell with applied forces.

$$\hat{Q}_x=K\left[-\frac{(1-\nu)}{2R^3}u_{,\theta\theta}+\frac{3-\nu}{2R^2}v_{,x\theta}\right.$$
$$\left.-\frac{2-\nu}{R^2}w_{,x\theta\theta}-w_{,xxx}\right] \qquad (5c)$$

$$M_x=K[(\nu/R^2)(v_{,\theta}-w_{,\theta\theta})-w_{,xx}] \qquad (5d)$$

Modal Functions

A general relation for the displacements in any mode may written[8,9] in the following form for any n

$$u(x,\theta,t)=\psi_u(x)\cos n\theta\sin\omega t$$
$$v(x,\theta,t)=\psi_v(x)\sin n\theta\sin\omega t$$
$$w(x,\theta,t)=\psi_w(x)\cos n\theta\sin\omega t \qquad (6)$$

where ψ_u, ψ_v, ψ_w are axial mode functions corresponding to axial, tangential, and radial displacements, respectively. The crucial part of the analysis is involved with choosing appropriate series forms for these axial mode functions. The series should be simple in form and preserve orthogonality properties. It is not necessary that the series satisfy any particular boundary conditions, since Lagrange multipliers can be used to enforce appropriate geometric boundary conditions. Because of the nature of the Rayleigh-Ritz energy procedure it is not necessary to enforce the natural boundary conditions.

There are two convenient sets of trigonometric series that meet all these requirements for the axial mode functions. The first set can be written in the form

$$\psi_u(x)\doteq A_{on}+\sum_{m=1}^{\infty}A_{mn}\cos\frac{m\pi x}{\ell} \qquad (7a)$$

$$\psi_v(x)=\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi x}{\ell} \qquad (7b)$$

$$\psi_w(x)=\sum_{m=1}^{\infty}C_{mn}\sin\frac{m\pi x}{\ell} \qquad (7c)$$

while the second set can be written as

$$\psi_u(x) = \sum_{m=1}^{\infty} A_{mn}\sin\frac{m\pi x}{\ell} \tag{8a}$$

$$\psi_v(x) = B_{on} + \sum_{m=1}^{\infty} B_{mn}\cos\frac{m\pi x}{\ell} \tag{8b}$$

$$\psi_w(x) = C_{on} + \sum_{m=1}^{\infty} C_{mn}\cos\frac{m\pi x}{\ell} \tag{8c}$$

The first set or CSS set represents the exact solution to the problem of an unconstrained shell with simply supported ends with no axial constraint. The SNA shell has boundary conditions at each end of the form

$$N_x=0 \quad v=0 \quad w=0 \quad M_x=0 \tag{9}$$

The second set or SCC set represents the exact solution to the problem of an unconstrained shell with freely supported ends with no tangential constraint. The FSNT shell has boundary conditions at each end of the form

$$u=0 \quad \hat{N}_{x\theta}=0 \quad \hat{Q}_x=0 \quad \partial w/\partial x=0 \tag{10}$$

It must be pointed out that, by using Lagrange multipliers in conjunction with Stokes' transformation, it is possible to use these sets to find exact solutions for shell problems with any possible combination of homogeneous end conditions. There are 16 possible sets of homogeneous boundary conditions that can be specified independently at each end[10] and these are listed in Table 1.

Stokes Transformation

When differentiating the foregoing series, one must take care with respect to the end values. For example, when a sine series is used to represent a function, the end values of the function are forced to be zero. With Stokes' transformation,[6,11] however, the end values of the sine series are released by being defined separately and these values are then included in the successive derivatives of the series.

Consider a function $f(x)$ represented by a Fourier sine series in the open range $0<x<\ell$ and by values f_0 and f_ℓ at the end points,

$$f(0)=f_0, \quad f(\ell)=f_\ell \tag{11a}$$

$$f(x) = \sum_{m=1}^{\infty} a_m\sin\frac{m\pi x}{\ell} \quad (0<x<\ell) \tag{11b}$$

Since it is not certain that the derivative $f'(x)$ can be represented by term by term differentiation of the sine series, the derivative is instead represented by an independent cosine series of the form

$$f'(x) = b_o + \sum_{m=1}^{\infty} b_m\cos\frac{m\pi x}{\ell} \tag{12}$$

Stokes' transformation then consists of integrating by parts in the basic definitions of the coefficients to obtain the required relationship between b_m and a_m. Similar care must be taken when finding the correct sine series corresponding to $f''(x)$. The complete set of formulas for the sine series is then written as

$$f(0)=f_0, \quad f(\ell)=f_\ell \tag{13a}$$

$$f(x) = \sum_{m=1}^{\infty} a_m\sin\frac{m\pi x}{\ell} \quad (0<x<\ell) \tag{13b}$$

$$f'(x) = -\frac{f_0-f_\ell}{\ell} - \sum_{m=1}^{\infty}\left[\frac{2}{\ell}\{f_0-(-1)^m f_\ell\} - \frac{\pi}{\ell}\, m\, a_m\right]\cos\frac{m\pi x}{\ell} \quad (0\le x\le\ell) \tag{13c}$$

$$f''(0)=f''_0 \quad f''(\ell)=f''_\ell \tag{13d}$$

Table 1 Boundary conditions of cylindrical shell

Case	Code	Description	Boundary conditions			
1	C	Clamped	u	v	w	$\partial w/\partial x$
2	CNA	Clamped with no axial constraints	N_x	v	w	$\partial w/\partial x$
3	CNT	Clamped with no tangential constraints	u	$\hat{N}_{x\theta}$	w	$\partial w/\partial x$
4	CNAT	Clamped with no axial, tangential constraints	N_x	$\hat{N}_{x\theta}$	w	$\partial w/\partial x$
5	SS	Simply supported	u	v	w	M_x
6	SNA	Simply supported with no axial constraints	N_x	v	w	M_x
7	SNT	Simply supported with no tangential constraints	u	$\hat{N}_{x\theta}$	w	M_x
8	SNAT	Simply supported with no axial, tangential constraints	N_x	$\hat{N}_{x\theta}$	w	M_x
9	FS	Freely supported	u	v	$\hat{Q}_x$	$\partial w/\partial x$
10	FSNA	Freely supported with no axial constraints	N_x	v	$\hat{Q}_x$	$\partial w/\partial x$
11	FSNT	Freely supported with no tangential constraints	u	$\hat{N}_{x\theta}$	$\hat{Q}_x$	$\partial w/\partial x$
12	FSNAT	Freely supported with no axial, tangential constraints	N_x	$\hat{N}_{x\theta}$	$\hat{Q}_x$	$\partial w/\partial x$
13	FAT	Free with axial, tangential constraints	u	v	$\hat{Q}_x$	M_x
14	FT	Free with tangential constraints	N_x	v	$\hat{Q}_x$	M_x
15	FA	Free with axial constraints	u	$\hat{N}_{x\theta}$	$\hat{Q}_x$	M_x
16	F	Free	N_x	$\hat{N}_{x\theta}$	$\hat{Q}_x$	M_x

$$f''(x)=(\frac{\pi}{\ell})\sum_{m=1}^{\infty}[\frac{2}{\ell}\{f_0-(-1)^m f_\ell\}m$$
$$-(\frac{\pi}{\ell})m^2 a_m]\sin\frac{m\pi x}{\ell}\quad(0<x<\ell)\qquad(13e)$$

If the function $f(x)$ is expanded in a Fourier cosine series, similar transformation formulas must be used to establish the correct form of the differentiated series. These formulas for the sine and cosine series are used in the Rayleigh-Ritz energy procedure to establish an explicit relation for strain energy, including end effects.

Application to Shell Problems

The preceding theoretical considerations will now be used to solve several cylindrical shell problems. The first set [Eqs. (7)] of axial mode functions, CSS, satisfies all the boundary conditions (both geometrical and natural) identically, only for the SNA shell with end conditions of simple support without axial constraint. For all other boundary conditions, Lagrange multipliers and Stokes' transformation must be used to enforce and release, respectively, appropriate geometric boundary conditions. Using CSS, the most complicated case (from the point of view of the size of the resulting frequency determinant) involves boundary conditions that are freely supported without tangential constraint at both ends (FSNT). It is interesting to note that, for the second set, SCC, all the boundary conditions of the FSNT shell are satisfied on a term by term basis, and this problem is, therefore, simple to solve. The most complicated problem for SCC is the SNA shell. In Table 2, an outline is given of the best set of axial mode functions to use to obtain the simplest frequency equation.

In the following work, details are only given for the solution using the CSS set. However, results for frequencies and mode shapes are shown based on both sets of axial mode functions. Further details involving both CSS and SCC are available in Ref. 7. For maximum generality of the formulation the FSNT shell will serve as the base problem for CSS. All other shell problems with different boundary conditions are simplified special cases of this base problem. For this base problem, four Lagrange multipliers are necessary to enforce geometrical boundary conditions and four unspecified end values are needed to release unwanted geometrical boundary conditions. The solution leads to an eight-by-eight frequency determinant. Solutions for shells with other boundary conditions lead to a smaller size determinant. It is well worth repeating at this point that, in practice, the simplest way to solve the FSNT shell is with the SCC set of functions; this leads to only a three-by-three determinant.

Table 2 Shell geometrical boundary conditions to be altered on using the first (CSS) or the second (SCC) set

		First set (CSS)		Second set (SCC)		
Case	Code	Forced to zero	Released	Forced to zero	Released	Better choice
1	C	$u\ \partial w/\partial x$	....	$v\ w$	....	1, 2
2	CNA	$\partial w/\partial x$	....	$v\ w$	u	1
3	CNT	$u\ \partial w/\partial x$	v	w	....	2
4	CNAT	$\partial w/\partial x$	v	w	u	1, 2
5	SS	u	....	$v\ w$	$\partial w/\partial x$	1
6	SNA	....	....	$v\ w$	$u\ \partial w/\partial x$	1
7	SNT	u	v	w	$\partial w/\partial x$	1, 2
8	SNAT	....	v	w	$u\ \partial w/\partial x$	1
9	FS	$u\ \partial w/\partial x$	w	v	....	2
10	FSNA	$\partial w/\partial x$	w	v	u	2
11	FSNT	$u\ \partial w/\partial x$	$v\ w$	....	....	2
12	FSNAT	$\partial w/\partial x$	$v\ w$	....	u	2
13	FAT	u	w	v	$\partial w/\partial x$	2
14	FT	....	w	v	$u\ \partial w/\partial x$	1
15	FA	u	$v\ w$	....	$\partial w/\partial x$	2
16	F	....	$v\ w$	....	$u\ \partial w/\partial x$	2

For the constrained shell (i.e., with intermediate supports) one Lagrange multiplier is necessary for each one of the constraint conditions to be enforced. A typical constrained shell with an axisymmetric constraint at $x=\xi$ is shown in Fig. 1. If this intermediate support has zero u, v, w constraints, then the maximum possible frequency determinant (for the FSNT shell with CSS set) is 11 by 11. In the following work it will be shown that it is a simple matter to extend the method to include as many intermediate supports as desired. In addition, these supports can have slope, as well as displacement, constraints.

A solution for the vibratory characteristics of the FSNT shell with displacement constraint at ξ_1 will now be outlined using CSS. The CSS set is given in Eqs. (7) and the boundary conditions appropriate to the FSNT shell are listed in Eqs. (10). In accordance with the variational principal the boundary conditions $\hat{N}_{x\theta}=0$ and $\hat{Q}_x=0$ need no explicit consideration since they are natural boundary conditions. The geometric boundary conditions that must be forced to zero are associated with u and $\partial w/\partial x$. Hence for arbitrary n it follows that

$$\psi_u(0)=A_{on}+\sum_{m=1}^{\infty}A_{mn}=0\qquad(14a)$$

$$\psi_u(\ell)=A_{on}+\sum_{m=1}^{\infty}A_{mn}(-1)^m=0\qquad(14b)$$

$$\psi_w'(0)=\frac{\pi}{\ell}[\frac{D_3+D_4}{2}+\sum_{m=1}^{\infty}(D_3+(-1)^m D_4$$
$$+m\,C_{mn})]=0\qquad(14c)$$

$$\psi_w'(\ell)=\frac{\pi}{\ell}[\frac{D_3+D_4}{2}+\sum_{m=1}^{\infty}(D_3+(-1)^m D_4$$
$$+m\,c_m)\ (-1)^m]=0\qquad(14d)$$

The last two formulas are obtained through the use of Stokes' transformation [Eqs. (11)] in which D_1-D_4, the unspecified end values, are defined in terms of the end values of ψ_v and ψ_w as

$$D_1=-(2/\pi)\psi_v(0)\quad D_2=(2/\pi)\psi_v(\ell)\qquad(15a)$$

$$D_3=-(2/\pi)\psi_w(0)\quad D_4=(2/\pi)\psi_w(\ell)\qquad(15b)$$

The constraint condition associated with the intermediate support of $u=v=w=0$ at $x=\xi_1$ are

$$\psi_u(\xi_1)=A_{on}+\sum_{m=1}^{\infty}A_{mn}\cos\frac{m\pi\xi_1}{\ell}=0\qquad(16a)$$

$$\psi_v(\xi_1)=\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi\xi_1}{\ell}=0\qquad(16b)$$

$$\psi_w(\xi_1)=\sum_{m=1}^{\infty}C_{mn}\sin\frac{m\pi\xi_1}{\ell}=0\qquad(16c)$$

The variational procedure associated with this problem then requires the following functional Φ to be made stationary

$$\Phi=(U_{\max}-T_{\max})-\lambda_1\psi_u(0)-\lambda_2\psi_u(\ell)-\lambda_3\psi_w'(0)$$
$$-\lambda_4\psi_w'(\ell)-\alpha_1\psi_u(\xi_1)-\beta_1\psi_v(\xi_1)-\gamma_1\psi_w(\xi_1)\qquad(17)$$

where the λ, $\alpha_1,\beta_1,\gamma_1$ are Lagrange multipliers. These Lagrange multipliers are related to the force type quantities required to enforce the constraint conditions. For example λ_1 and λ_2 may be interpreted as quantities proportional to the in-plane forces N_x that maintain the zero axial displacement at the ends ($u=0$ at $x=0, \ell$). The Langrange multipliers λ_3 and λ_4 are proportional to the bending moments M_x that maintain the zero radial slope condition at the ends ($w,_x=0$). Finally $\alpha_1, \beta_1, \gamma_1$ are proportional to the forces $N_x, N_{x\theta}$, and Q_x that maintain zero displacements at the intermediate support $x=\xi_1$.

An explicit relation for $U_{\max}$ and $T_{\max}$ is obtained by substitution of the CSS set and its appropriate derivatives via Stokes' transformation into Eqs. (1) and (2)

$$U_{\max} = U_{\max}(A_{on}, A_{mn}, B_{mn}, C_{mn}, D_1, D_2, D_3, D_4) \tag{18a}$$

$$T_{\max} = T_{\max}(A_{on}, A_{mn}, B_{mn}, C_{mn}) \tag{18b}$$

Substitution of Eqs. (18) and the constraint relations Eqs. (14) and (16) into the functional Eq. (17) and variation with respect to A_{on}, A_{mn}, B_{mn}, C_{mn} (setting partial derivatives to zero) then leads to an explict relation for A_{on} and, in addition, a set of equations in which A_{mn}, B_{mn}, C_{mn} are coupled together. The solution of these coupled equations leads to all the coefficients to be written explicitly in terms of the 11 quantities λ_1, λ_2, λ_3, λ_4, D_1, D_2, D_3, D_4, α_1, β_1, γ_1 and the frequency parameter Ω.

The frequency determinant may now be easily constructed. From the constraint conditions and the stationary character of Φ 11 equations can be obtained. Four equations are obtained from Eqs. (14) representing the constraint conditions due to the geometric boundary conditions of zero axial displacement and zero radial slope at both ends. Variation of Φ with respect to the four unspecified end values D_1, D_2, D_3, D_4 leads to four equations associated with the natural boundary conditions. The two equations obtained from D_1, D_2 are identical to the natural boundary condition $\hat{N}_{x\theta}=0$ at the ends $x=0,\ell$. The two equations obtained from D_3, D_4 may be shown to be identical to the result obtained from the natural boundary condition $\hat{Q}_x=0$. Finally, three equations are obtained from Eqs. (16) representing the intermediate constraint condition of zero displacement at $x=\xi_1$.

Substitution of the relations for the coefficients A_{on}, A_{mn}, B_{mn}, C_{mn} into the above 11 conditions leads to the homogeneous matrix equation

$$[e_{i,j}]\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, D_1, D_2, D_3, D_4, \alpha_1, \beta_1, \gamma_1\}^T = \{0\} \tag{19}$$

where $i,j=1,2,\ldots,11$. For the nontrivial solution the determinant of the matrix is zero. This leads to the following exact symmetric frequency determinant, the elements of which are infinite series. The natural

$$|e_{i,j}| = 0 \tag{20}$$

frequency appears as a parameter $\Omega=(\omega/\omega_0)^2$ in this equation. The complete development of this frequency determinant is given in Ref. 7. From the development of these equations it may be shown that there are two restrictions involved. First restriction is $\Omega \neq (1-\nu)/2)\ (1+k/4)n^2$. For axisymmetric ($n=0$) shell vibrations, this becomes $\Omega \neq 0$; thereby invalidating Eq. (20) for rigid body motion. However, conditions associated with this rigid body motion are usually determined by inspection. For nonaxisymmetric motion ($n \neq 0$), the case of $\Omega=(1-\nu)/2)\ (1+k/4)n^2$ is automatically rejected on the grounds that it leads to rigid body motion with nonzero frequency. The second restriction may be shown to imply that this CSS set cannot be used to solve the unconstrained FSNT shell. This is not a series drawback since it is easy to show that the SCC set offers a very simple solution to this case.

Frequency Determinant

Unconstrained Shell

The frequency determinant is derived here for several cases of shells without intermediate constraint. The simplest problem is, of course, the SNA shell since this CSS set of axial mode functions satisfies all the boundary conditions (both geometric and natural) identically. Thus, the present Rayleigh-Ritz procedure leads to the exact frequency determinant without any modification associated with Lagrange multipliers or Stokes' transformation. The elements of the frequency determinant are no longer infinite series but reduce to algebraic terms. Past work on this problem includes that of Arnold and Warburton[8] and Dym.[12]. For any fixed n, the frequency determinant yields three natural frequencies. Each of these frequencies has a different modal configuration associated with it. For the axisymmetric case of $n=0$, this frequency determinant checks against Eqs. (4a) and (4b) of Ref. 4.

As a typical case involving the enforcement of geometric boundary conditions, consider the shell clamped at both ends. The boundary conditions are

$$u=0 \quad v=0 \quad w=0 \quad \partial w/\partial x=0 \quad (x=0,\ell) \tag{21}$$

The CSS set requires a total of four Lagrange multipliers to enforce at each end the geometric boundary conditions $u=0$ and $\partial w/\partial x=0$, which are not automatically satisfied by this set. The frequency determinant is then found from Eq. (20) by retaining the rows and columns associated with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The end result is the exact four-by-four frequency determinant

$$\begin{vmatrix} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} \\ & e_{2,2} & e_{2,3} & e_{2,4} \\ \text{symmetric} & & e_{3,3} & e_{3,4} \\ & & & e_{4,4} \end{vmatrix} = 0 \tag{22}$$

It is interesting to note that the second set, SCC, would also lead to a (different) four-by-four determinant.

As a typical case involving the release of unwanted geometrical boundary conditions consider a free-free shell with boundary conditions

$$N_x=0 \quad \hat{N}_{x\theta}=0 \quad \hat{Q}_x=0 \quad M_x=0 \quad (x=0,\ell) \tag{23}$$

From the properties of CSS it is apparent that the tangential and radial displacements, v and w, respectively, are identically zero at the ends. Therefore, the releasing procedure is required to remove these unwanted geometric boundary conditions. By defining v and w separately at the end points [Eqs. (15)] and including these values in the subsequent differentiation via Stokes' transformation, the exact frequency equation can be found. The result is obtained from Eq. (20) by retaining the rows and columns associated with $D_1 - D_4$

$$\begin{vmatrix} e_{5,5} & e_{5,6} & e_{5,7} & e_{5,8} \\ & e_{6,6} & e_{6,7} & e_{6,8} \\ \text{symmetric} & & e_{7,7} & e_{7,8} \\ & & & e_{8,8} \end{vmatrix} = 0 \tag{24}$$

It is apparent that the two techniques of forcing and releasing appropriate geometric boundary conditions can be combined by simultaneously employing Lagrange multipliers and Stokes' transformation. A practical problem involving this procedure is the clamped-free shell. This cantilevered shell has been studied by Sharma and Johns[13] using beam eigenfunctions in the Rayleigh-Ritz procedure and by Warburton and Higgs[14] using a direct solution to the shell equations. For this nonsymmetric set of boundary conditions

$$u=0 \quad v=0 \quad w=0 \quad \partial w/\partial x=0 \quad (x=0) \tag{25a}$$

$$N_x=0 \quad \hat{N}_{x\theta}=0 \quad \hat{Q}_x=0 \quad M_x=0 \quad (x=\ell) \tag{25b}$$

the CSS set leads to a four-by-four frequency determinant. The $u=0$ and $\partial w/\partial x=0$ conditions at $x=0$ must be enforced with two Lagrange multipliers. The $v=0$ and $w=0$ conditions at $x=\ell$ must be released using Stokes' transformation. Retaining the rows and columns in Eq. (20) associated with $\lambda_1, \lambda_3, D_2, D_4$ leads to

$$\begin{vmatrix} e_{1,1} & e_{1,3} & e_{1,6} & e_{1,8} \\ & e_{3,3} & e_{3,6} & e_{3,8} \\ \text{symmetric} & & e_{6,6} & e_{6,8} \\ & & & e_{8,8} \end{vmatrix} = 0 \tag{26}$$

Constrained Shell

Consider the constrained shell with an intermediate constraint at $x=\xi_I$. Intermediate constraint conditions on displacement are enforced by using the Lagrange multipliers $\alpha_I, \beta_I, \gamma_I$ [Eq. (17)]. Extra intermediate supports are input with additional Lagrange multipliers. Consider the previous four-by-four frequency determinant for the clamped-free shell. If an intermediate constraint is placed as $x=\xi_I$ with the condition of $u=v=w=0$ then the frequency equation becomes the following seven-by-seven with the extra rows and columns associated with the Lagrange multipliers $\alpha_I, \beta_I, \gamma_I$

$$\begin{vmatrix} e_{1,1} & e_{1,3} & e_{1,6} & e_{1,8} & e_{1,9} & e_{1,10} & e_{1,11} \\ & e_{3,3} & e_{3,6} & e_{3,8} & e_{3,9} & e_{3,10} & e_{3,11} \\ & & e_{6,6} & e_{6,8} & e_{6,9} & e_{6,10} & e_{6,11} \\ & & & e_{8,8} & e_{8,9} & e_{8,10} & e_{8,11} \\ \text{symmetric} & & & & e_{9,9} & e_{9,10} & e_{9,11} \\ & & & & & e_{10,10} & e_{10,11} \\ & & & & & & e_{11,11} \end{vmatrix} = 0 \tag{27}$$

It is interesting to note that for the axisymmetric mode ($n=0$), the torsional vibration associated with the circumferential displacement uncouples from the coupled radial and axial vibration. This may be noted by verifying that the coefficients B_{m0} of the series for the tangential displacement v are uncoupled from the other coefficients A_{m0}, C_{m0}. Another way of verifying this uncoupling result is to examine directly the frequency determinant for this case of $n=0$. It should be carefully noted that, unlike the $n=0$, for the case of $n \geq 1$ this simplification of the frequency determinant due to the uncoupling of the tangential motion from the radial and longitudinal motion no longer occurs.

Results and Conclusions

Frequency results were obtained from the frequency determinant by substituting in values of ω/ω_0 and monitoring the determinant until it vanished. Before this analysis was done, manipulations were performed to make the rate of convergence of each infinite series term as close as possible to each other. The convergence of certain terms was accelerated by subtracting off a series with a known sum, the terms of which are asymptotically equivalent to the original series. In this way fast convergence was obtained and the same upper limit could be used (typically so). Results are presented in dimensionless form for both frequency and mode shape.

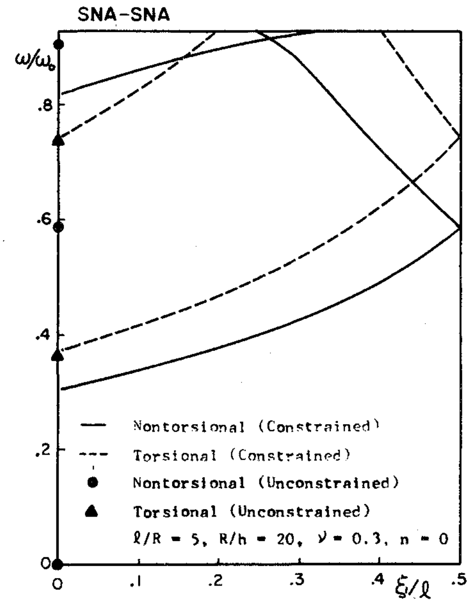

Fig. 2 Frequency spectrum for constrained ($u=v=w=0$ at $x=\xi_I$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=0$.

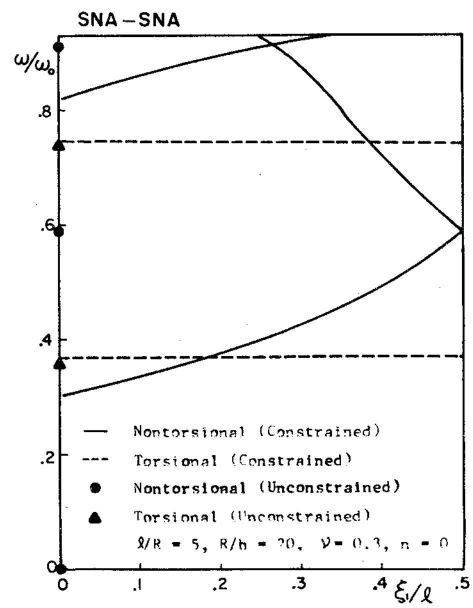

Fig. 3 Frequency spectrum for constrained ($u=w=0$ at $x=\xi_I$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=0$.

Results for *constrained* shells are presented for a wide variety of boundary conditions and intermediate constraint positions. For numerical results, the shell parameters used are $\ell/R=5$, $R/h=20$, $\nu=0.3$. Results are shown for shells with intermediate constraints $u=v=w=0$ as well as $u=w=0$. For cases with symmetric boundary conditions, frequency plots are only shown up to $\xi_I/\ell=0.5$ since the frequencies are symmetric about this central position.

In Figs. 2-5 are shown frequency and mode shape results for the axisymmetric ($n=0$) mode of vibration. Some discussion of the frequency spectrum for this $n=0$ case with SNA boundary conditions was given in Ref. 4. As mentioned previously for axisymmetric vibration, the torsional modes are always coupled from the coupled radial and axial modes. Therefore, torsional vibrations are entirely dependent on the boundary conditions and intermediate constraint conditions associated with the tangential displacement v of the shell. For the lowest nontorsional mode (coupled axial and radial motion) the mode shape is predominately longitudinal in

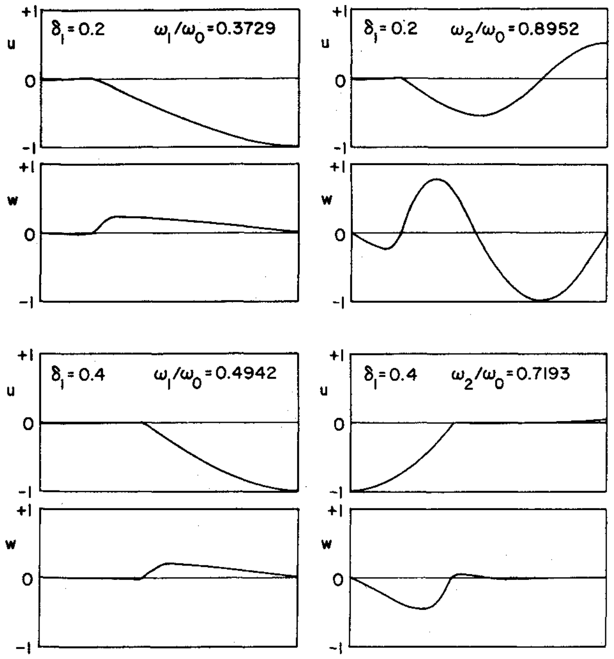

Fig. 4 Mode shapes for constrained ($u=v=w=0$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=0$ ($\ell/R=5$, $R/h=20$, $\nu=0.3$, nontorsional motions).

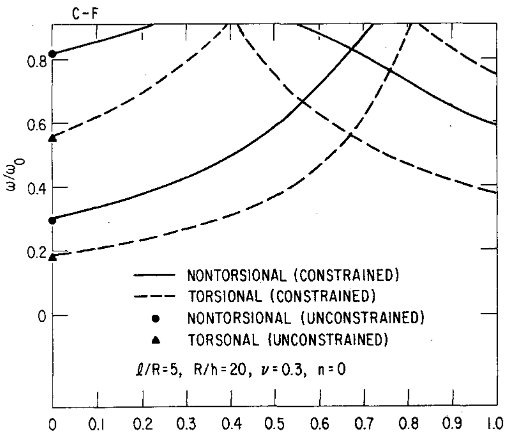

Fig. 5 Frequency spectrum for constrained ($u=v=w=0$ at $x=\xi_1$) shell with clamped-free ends (C-F) for $n=0$.

character. Results are plotted for the SNA shell in Figs. 2-4. The omission of the v constraint at the intermediate constraint in Fig. 3 leads to the torsional frequency ($\omega/\omega_0=0.372$) being independent of intermediate constraint position ξ_1. It is interesting to note in Fig. 3 that, depending on the position of the intermediate constraint, the lowest frequency may be associated with torsional motion or with coupled radial-axial motion. The mode shape diagrams for nontorsional motion are particularly enlightening (Fig. 4). For the lowest frequency in the nontorsional mode it is shown that the axial motion predominates. Consequently, this mode may be qualitatively compared to the lowest mode of a bar fixed at one end (corresponding to the constraint position in the shell) and free at the other end with the equivalent length being the larger of either ξ_1 or $\ell-\xi_1$. This bar model is an aid in the explanation of why in Fig. 4 for $\delta_1=\xi_1/\ell=0.2$ the second mode has motion predominately to the right of the support, while for $\delta=0.4$ the motion in the second mode takes place predominately to the left of the support. The frequency spectrum for the *cantilevered* shell with $u=v=w=0$ constraint at $x=\xi_1$ is shown in Fig. 5. This spectrum is no longer symmetric about the central position since the boundary conditions are not symmetric. The lowest branch of the frequency spectrum is associated with the uncoupled torsional mode, and represents the torsional frequencies of either a clamped-clamped shell with length ξ_1 or a clamped-free shell with length $\ell-\xi_1$. The next branch of the spectrum represents the coupled radial and axial mode. The frequencies may be closely approximated by the lowest nontorsional frequency of either a clamped-clamped shell with length ξ_1 or a clamped-free shell with length $\ell-\xi_1$. Once again the motion in this mode is primarily axial. It is interesting to note that an optimum constraint position (to maximize the frequencies in this first nontorsional mode) is about $\xi_1/\ell=0.67$, which is the nodal point of the second mode of longitudinal vibration of a fixed-free bar with length ℓ.

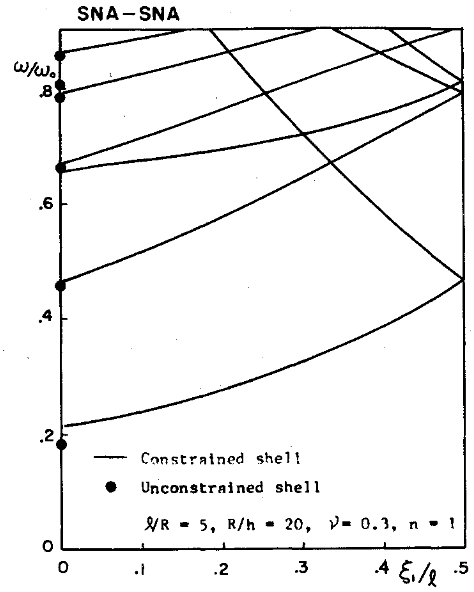

Fig. 6 Frequency spectrum for constrained ($u=v=w=0$ at $x=\xi_1$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=1$.

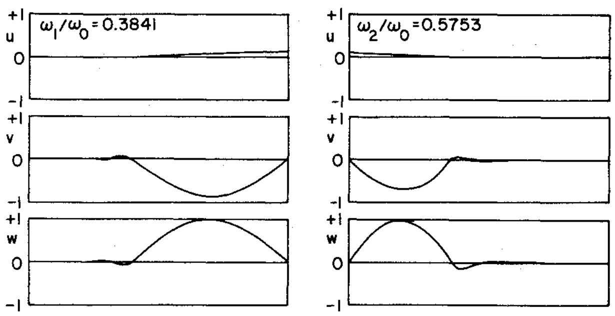

Fig. 7 Mode shapes for constrained ($u=v=w=0$ at $\xi_1/\ell=0.4$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=1$.

Frequencies and modes for the $n=1$ case are shown in Figs. 6-7. For this beam-type motion all three displacements u, v, w are coupled together; the analysis is, therefore, more complicated than for the axisymmetric case. For this mode, tangential constraints have a strong influence on the frequency characteristics. In Figs. 6 and 7 are shown frequency and mode shape data for the SNA shell. For the shell without intermediate constraint, the first two natural frequencies are found to be $\omega/\omega_0=0.1864$, 0.4595. However, the first two frequencies for the constrained shell, as the $u=v=w=0$ constraints approach the boundary, are $\omega/\omega_0=0.2122$, 0.4640. The discrepancy is due to the difference in the longitudinal constraint conditions for the two shells (the unconstrained shell has $N_x=0$ at $x=0$ but the constrained shell boundary condition approaches $u=0$ at $x=0$). The mode shape diagram shows that once again basically one portion of the shell is in

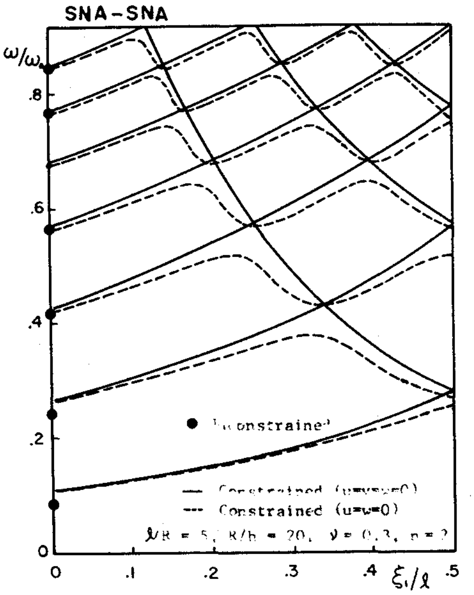

Fig. 8 Frequency spectrum for constrained ($u=v=w=0$ or $u=w=0$ at $x=\xi_I$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=2$.

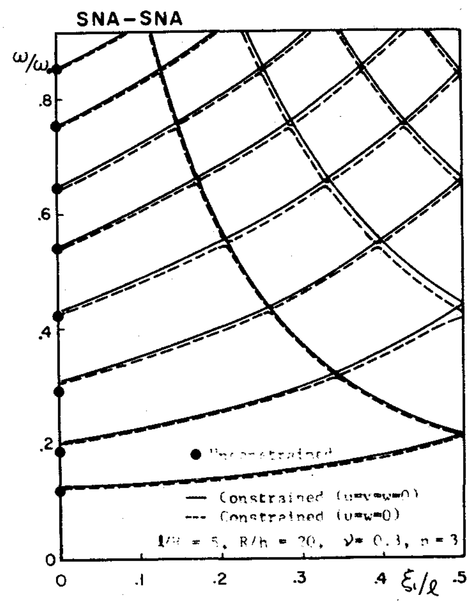

Fig. 9 Frequency spectrum for constrained ($u=v=w=0$ or $u=w=0$ at $x=\xi_I$) shell with simply supported ends without axial constraints (SNA-SNA) for $n=3$.

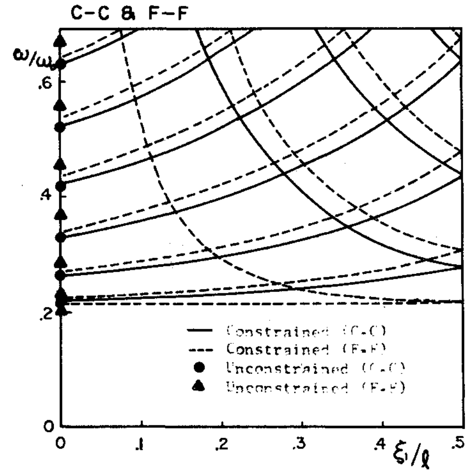

Fig. 10 Frequency spectrum for constrained ($u=v=w=0$ at $x=\xi_I$) shell with clamped (C-C) or free ends (F-F) for $n=4$.

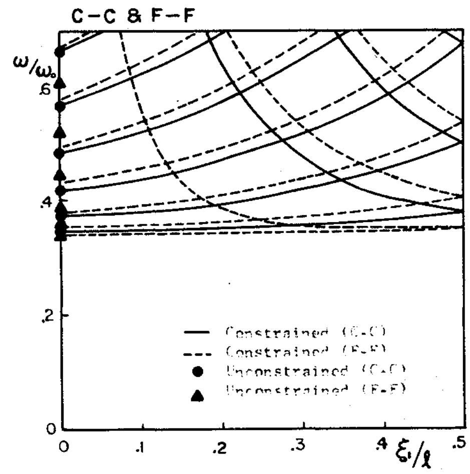

Fig. 11 Frequency spectrum for constrained ($u=v=w=0$ at $x=\xi_I$) shell with clamped (C-C) or free ends (F-F) for $n=5$.

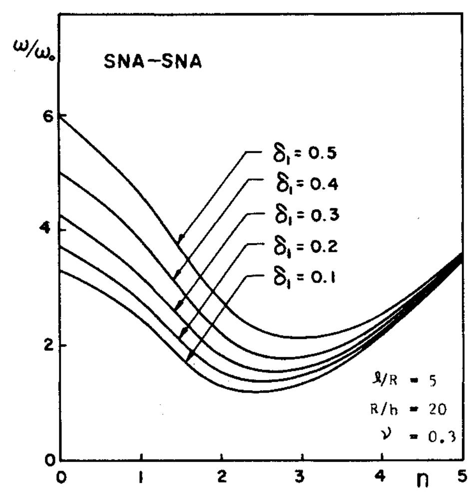

Fig. 12 Lowest frequency parameters, ω/ω_0 for constrained ($u=v=w=0$ at $x=\xi_I$) shell with simply supported ends without axial constraints (SNA-SNA).

motion. For $\xi_I/\ell=0.4$ the lowest mode, $\omega/\omega_0=0.3841$ responds essentially over the 0.6ℓ portion while in the second mode, $\omega/\omega_0=0.5753$, the response is basically in the 0.4ℓ portion.

As the circumferential node number becomes larger ($n\geq 2$) localized behavior becomes more important and the influence of both boundary conditions and intermediate constraints tends to diminish. Frequency curves are shown in Figs. 8 and 9 for the SNA shell with $u=v=w=0$, as well as $u=w=0$ constraint at $x=\xi_I$. For $n=2$ there are still significant differences in the frequencies associated with these two types of constraint, but for $n=3$ these differences are much smaller. For $n=4$, 5 these differences essentially disappear.[7] In addition, the bottom branch of the frequency spectrum tends to flatten out with increasing n, implying that the results are tending to become independent of constraint position. For $n=4$, 5, results are presented in Figs. 10 and 11 for the C-C and F-F shell. Similar qualitative comparisons can be made to the aforementioned trends for the SNA shell. The flattening of the lowest branch is seen; the lowest frequency for either C-C or F-F shell with $n=5$ lies between $\omega/\omega_0=0.34$-0.38.

The lowest frequency parameter is plotted against mode number in Figs. 12 and 13 for the SNA and C-C shells, respectively. The lowest frequency occurs at either $n=2$ or $n=3$,

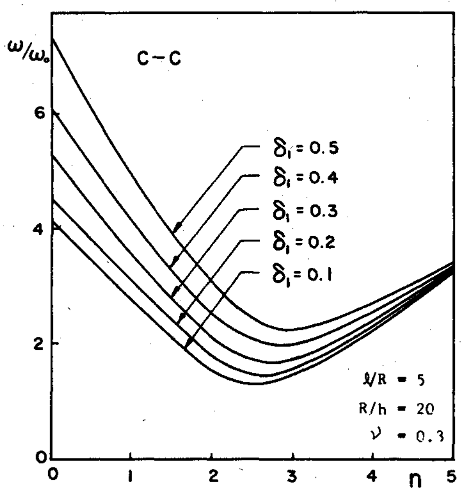

Fig. 13 Lowest frequency parameters, ω/ω_0 for constrained ($u=v=w=0$ at $x=\xi_1$) shell with clamped ends (C-C).

depending on the boundary conditions and the intermediate constraint position. In addition, it can clearly be seen that the lowest frequency becomes independent of intermediate constraint position as n becomes large.

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